SAJSET
ISSN: 2992-3581

Savannah Journal of Science and Engineering Technology
Volume: 03; Issue: 06; Pages: 609-614; 2025.
www.sajsetjournal.com.ng

DOI: xx.xxxx.x/sajset-03-2025-0049

SAJSET
ISSN: 2992-3581

Hybridization techniques for the solution of system of nonlinear equations using conjugate gradient method

[1]Bulama, S., [1]Iggi, N. A., [1]Adam, M. I., [2]Mohammed, R. and [2]Abbas, A.

[1]*Department of General Studies, Jigawa State Polytechnic for Information and Communication Technology Kazaure, Jigawa State, Nigeria*

[2]*Department of Management Sciences, Jigawa State College of Remedial and Advanced Studies, Babura, Jigawa State, Nigeria*

nuraiggi82@gmail.com, mibrahim.sta@jspict.edu.ng, rabiujicora@gmail.com, abbasabdul21@gmail.com

Paper History

Received: 10th Oct., 2025
Accepted: 02nd Nov., 2025
Published: November, 2025

Corresponding author
Bulama, S.
seedbulama@gmail.com

Abstract:

The conjugate gradient method for system of nonlinear equations is rare as most of the methods are for unconstrained optimization and may not necessarily generate sufficient descent or descent direction. Most of the current studies are on the New Hybrid Algorithm for Convex Nonlinear Unconstrained Optimization. This paper presents an effective conjugate gradient method via hybridization techniques for the solution of system of nonlinear equations. The study aimed to use hybridization techniques to solve system of nonlinear equations using conjugate gradient methods. And the study is to achieve these objectives; to derive methods that are derivative free, to present an effective conjugate gradient method, to present methods with sufficient descent direction. We achieved these by hybridizing some well-known conjugate gradient methods. The scheme satisfies the sufficient decent condition. Under mild condition, the global convergence result for the method is established. Preliminary numerical results for some large-scale benchmark test problems reported in this work, demonstrate that, the method is practically effective and competitive to some existing methods. This study have applications in engineering such as structural engineering, mechanical engineering, electrical and electronic engineering, applied mathematics and numerical analysis, computer science and machine learning etc. in solving system of nonlinear equations using the conjugate gradient method which is fast and efficient.

Keywords: *Conjugate gradient, Nonlinear equations, Hybridization*

1. Introduction

Consider the general form of system of nonlinear equations as shown in equation 1 [1]:

$$F(x) = 0 \qquad (1)$$

Where $F: \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map which assumed to be continuously differentiable functions. The system of nonlinear equations arises in many areas of scientific computing and engineering applications. A variety of different iterative methods have been developed for solving problem (1), for example, Newton's method, quasi-Newton method, Gauss-Newton Method [1] and their variants. However, they are not particularly suitable for solving large-scale problems, because they need to solve linear system of equations using Jacobian matrix or its approximation at each iteration. It is vital to mention that, the conjugate gradient (CG) methods are among the popular methods used to solve large-scale system of nonlinear equations, due to their rapid convergence property, simple to implement and low storage requirement [2, 3]. In fact, conjugate gradient method has played a vital role in solving optimization problems.

However, it generates a sequence of iterative points $\{x_k\}$ from an initial guess $x_0 \in \mathbb{R}^n$, using the iterative equation 2 reported in Can [4]

$$x_{k+1} = x_k + \alpha_k d_k, \qquad k = 0,1,2, \ldots \qquad (2)$$

Where x_k is the previous iterative point, x_{k+1} is the current iterative point, $\alpha_k > 0$ is the step-length computed via any suitable line search technique and d_k is the search direction defined by equation 3:

$$d_k = \begin{cases} -F(x_k), & if \ \ k = 0, \\ -F(x_k) + \beta_k d_{k-1}, & if \ \ k \geq 1 \end{cases} \qquad (3)$$

Where $F_k = F(x_k)$ and β_k is termed as conjugate gradient parameter. Conjugate gradient methods differ in their way of defining the CG update parameter β_k because different choices of β_k give rise to distinct conjugate gradient methods with quite different computational efficiency and convergence properties.

Moh'd, *et al.* [5] presents a Hybrid Broyden-Fletcher-Goldfab-Shanno (HBFGS) method which used the search direction of the conjugate gradient methods with quasi-Newton update where their numerical result provides strong evidence that the proposed HBFGS method is more efficient than the ordinary BFGS method. Furthermore, Mustapha, *et al.* [6] followed the approach in Mohammad, [7] to present a Hybrid BFGS CG method for solving unconstrained optimization problems; the method has been presented based on combining search directions between conjugate gradient method and quasi-Newton method where the methods have shown some significant improvement for solving large-scale problems with less number of iterations and CPU time respectively. However, Hamed, *et al.* [8] equally modified the work in Mustapha, *et al*. [6] and Mohammad, *et al.* [7] and presented a new algorithm for convex Nonlinear Unconstrained optimization problems by proposing the search direction as defined in Mohammad, *et al.* (2014) [7] and Mustapha, *et al.* [6].The new search direction is defined as shown in equation 4:

$$d_{k+1} = -\lambda_k g_{k+1} + \rho_k H_{k+1} g_{k+1} \quad (4)$$

Where H_{k+1} is the approximation matrix of BFGS updated matrix and λ_k is a positive constant. And the parameters λ_k and ρ_k are respectively defined as shown in equations 5 and 6 respectively [6, 7]:

$$\lambda_k = \frac{(1+t)s_k^T g_{k+1}}{y_k^T g_{k+1}}, \quad (5)$$

and

$$\rho_k = \frac{\lambda_k y_k^T g_{k+1} - t s_k^T g_{k+1}}{s_k^T g_{k+1}}. \quad (6)$$

Where $t > 0$ and the conjugate gradient (CG) parameter is obtained as shown in equation 7 [6, 7]:

$$\beta_k = \frac{\psi_k \rho_k y_k^T g_{k+1} + s_k^T y_k g_{k+1} s_k^T - \lambda_k s_k^T y_k g_{k+1}^T s_k}{||s_k||^2 s_k^T y_k} \quad (7)$$

Therefore, it is very important to state that, solving BFGS-CG methods is severally used in unconstrained optimization problems, they are particularly efficient due to their rapid Convergence properties, simple to implement and low storage requirement [8, 9]. However, they are very scanty in solving system of nonlinear equations; this is what motivated us to write this paper. Furthermore, equation 1 can come from an unconstrained optimization problem, a saddle point and equality constrained problem [10]. Let f be a norm function defined by;

$$f(x) = \frac{1}{2}||F(x)||^2. \quad (8)$$

Then the nonlinear equation in problem (1) is equivalent to the following global optimization problem [9, 11].

$$min f(x), \quad x \in \mathbb{R}^n. \quad (9)$$

We organize the paper as follows. In the next section, we present the details of our proposed method, convergence result is presented in section 3. Some numerical results are reported in section 4. Finally, conclusions are made in section 5.

2. Derivation of the method

In this section, we present new hybrid conjugate gradient (CG) update parameter β_k, via two other parameters λ_k and ρ_k. This is made possible by combining the search direction proposed by Hamed et al. [8], given by equation 10:

$$d_{k+1} = -\lambda_k F(x_{k+1}) + \rho_k J_{k+1}^{-1} F(x_{k+1}), \quad (10)$$

Together with the classical Newton direction given by equation 11:

$$d_{k+1} = -J_{k+1}^{-1} F(x_{k+1}) + \lambda_k d_k \quad (11)$$

Where J_{k+1}^{-1} is the inverse Jacobian matrix. Multiplying equations 10 and 11 by y_k^T we have equations 12 to 15;

$$y_k^T d_{k+1} = -\lambda_k y_k^T F(x_{k+1}) + \rho_k y_k^T J_{k+1}^{-1} F(x_{k+1}), \quad (12)$$

$$y_k^T d_{k+1} = -y_k^T J_{k+1}^{-1} F(x_{k+1}) + \lambda_k y_k^T d_k. \quad (13)$$

By conjugacy condition;

$$y_k^T d_{k+1} = 0 \quad (14)$$

And also from weak secant condition, i.e

$$y_k^T J_{k+1}^{-1} = s_k^T \quad (15)$$

We assume J_{k+1}^{-1} is symmetric. By applying equation 14 and 15 in equation 12 we obtain equation 16:

$$\lambda_k = \frac{s_k^T F(x_{k+1})}{y_k^T d_k} \quad (16)$$

Also, substituting equations (14) and (15) in (12) we have equation 17;

$$\rho_k = \frac{\lambda_k y_k^T F(x_{k+1})}{s_k^T F(x_{k+1})} \quad (17)$$

Recall that the classical CG direction is defined to obtain an updated version of the conjugate gradient method associated with new parameter β_k, compare the standard CG direction;

$$d_{k+1} = -F(x_{k+1}) + \beta_k s_k, \quad (18)$$

with

$$d_{k+1} = -\lambda_k F(x_{k+1}) + \rho_k J_{k+1}^{-1} F(x_{k+1}). \quad (19)$$

Therefore, form (18) and (19) we have;

$$-\lambda_k F(x_{k+1}) + \rho_k J_{k+1}^{-1} F(x_{k+1}) = -F(x_{k+1}) + \beta_k s_k. \quad (20)$$

Multiplying (20) by y_k^T we have;

$$-\lambda_k y_k^T F(x_{k+1}) + \rho_k y_k^T J_{k+1}^{-1} F(x_{k+1}) = -y_k^T F(x_{k+1}) + \beta_k y_k^T s_k \quad (21)$$

After some algebraic simplifications, we obtain our proposed parameter β_k as;

$$\beta_k = \frac{\rho_k s_k^T F(x_{k+1}) + y_k^T F(x_{k+1}) - \lambda_k y_k^T F(x_{k+1})}{y_k^T s_k}. \quad (22)$$

Equation 22 further simplifies to:

$$\beta_k = \frac{\rho_k s_k^T F(x_{k+1}) + (1-\lambda_k) y_k^T F(x_{k+1})}{y_k^T s_k}. \quad (23)$$

Finally, substituting equations 16) and 17 in equation 23, the CG parameter β_k becomes:

$$\beta_k = \frac{y_k^T F(x_{k+1})}{y_k^T s_k}, \quad (24)$$

and our search direction is given by:

$$d_{k+1} = -F(x_{k+1}) + \left(\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right) s_k \quad (25)$$

2.1 Algorithm 1 (Hybrid Conjugate Gradient Algorithm)

Algorithm 1 **(SHCGA)**

Step 1: Given $x_0 \in \mathbb{R}^n$, $\varepsilon > 0, d_0 = -F(x_0)$, $set\ \ k = 0$.
Step 2: Compute $F(x_k)$.
Step 3: If $||\ F(x_k)|| \le \varepsilon,\ then\ stop,\ else,\ go\ to\ step\ 4$.
Step 4: Compute the step-length α_k using equation 34.
Step 5: Set $x_{k+1} = x_k + \alpha_k d_k$.
Step 6: Compute $F(x_{k+1})$.
Step 7: Compute $y_k = F(x_{k+1}) - F(x_k)$.
Step 8: Compute $\lambda_k\ \ and\ \rho_k$ from equations 16 and 17.
Step 9: Compute β_k using equation 23.
Step 10: Update d_{k+1} using equation 27.
Step 11: Set $k = k + 1$ and go back to step 2.

2.2 Convergence Analysis

This section is devoted to a study of the global convergence of our method (SHCGA). To begin with, let us define the level set:

$$\Omega = \{x: \left||F(x)|\right| \le \left||F(x_0)|\right|\} \quad (35)$$

The following basic assumptions are required in order to analyze the convergence of our algorithm 1.

2.2.1 Assumptions:

a. There exists $x^* \in \mathbb{R}^n$ such that $F(\ x^*) = 0$.
b. F is continuously differentiable in a neighborhood of x^*.
c. The level set Ω as defined by equation 35 is bounded.
d. CG direction is a good approximation to Newton direction, i.e

$$\left||F'(x_{k+1})d_{k+1} - (d_{k+1} - \beta_k s_k)|\right| \le \varepsilon\left||F(x_{k+1})|\right|, \quad (36)$$

Where $\varepsilon\ \epsilon (0,1)$ is a small quantity in equation 12, 13 and 14.

e. The Jacobian of F is bounded and positive definite on N. i.e there exists positive constants $M > m > 0$ such that:

$$\left||F'(x)|\right| \le M \qquad \forall x \epsilon N,\ and \quad (37)$$

$$m\left||d^2|\right| \le d^T F'(x) d \qquad \forall x\ x\epsilon N, \quad d\epsilon\mathbb{R}^n. \quad (38)$$

Lemma 1 Suppose assumption 1 holds. Let $\{x_k\}$ be generated by the SHCGA algorithm, then

$$\lim_{k\to\infty} ||\alpha_k d_k\ || = \lim_{k\to\infty}\left||s_k|\right| = 0, \quad (39)$$

and

$$\lim_{k\to\infty} ||\alpha_k F(x_k)\ || = 0 \quad (40)$$

Proof: From the line search in equation 34 and for all $k > 0$, we obtain:

$$\omega_2\left||\alpha_k d_k|\right|^2 \le \omega_1\left||\alpha_k F_k|\right|^2 + \omega_2\left||\alpha_k d_k|\right|^2$$

$$\le ||F_k||^2 - ||F_{k+1}||^2 + \eta_k ||F_k||^2 \quad (41)$$

And by summing up the above k inequality, we obtain:

$$\omega_2 \textstyle\sum_{i=0}^{k}\left||\alpha_k d_k|\right|^2 \le \sum_{i=0}^{k}\left(\left||F(x_i)|\right|^2 - \left||F(x_{i+1})|\right|^2\right) + \sum_{i=0}^{k}\eta_i ||F(x_i)^2$$

$$= ||F(x_0)||^2 - ||F(x_{k+1})||^2 + \textstyle\sum_{i=0}^{k}\eta_k ||F(x_i)||^2$$

$$\le ||F(x_0)||^2 + ||F(x_0)||^2 \textstyle\sum_{i=0}^{k}\eta_i$$

$$\le ||F(x_0)||^2 + ||F(x_0)||^2 \textstyle\sum_{i=0}^{\infty}\eta_i$$

$$\le M^2 + M^2 \textstyle\sum_{i=0}^{\infty}\eta_i \quad (42)$$

Therefore, from the level set and the fact that $\{\eta_k\}$ satisfies equation 33, then the series $\sum_{i=0}^{k}||\alpha_k d_k||^2$ is convergent, which implies that equation 39 holds. Using the same argument as above, with $\omega_1\left||\alpha_k F_k|\right|^2$ on the left hand side, we obtain equation 40.

Lemma 2 Suppose assumption 1 holds. Let the sequence $\{x_k\}$ be generated by the SHCGA algorithm with update, then there exists a constant $m_2 > 0$ such that for k>0,

$$\left||d_k^{SHCGA}|\right| \le m_2 \quad (43)$$

Proof: From equation 27 we have

$$\left||d_{k+1}|\right| = \left|\left|-F(x_{k+1}) + \left(\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right) s_k\right|\right|. \quad (44)$$

Applying triangular inequality, we've;

$$\left||d_{k+1}|\right| \le \left||F(x_{k+1})|\right| + \left|\left|\frac{y_k^T F(x_{k+1})}{y_k^T s_k}\right|\right| ||s_k|| \quad (45)$$

$$\le \left||F(x_{k+1})|\right| + \frac{|y_k^T F(x_{k+1})|||s_k||}{y_k^T s_k} \quad (46)$$

$$\le \left||F(x_{k+1})|\right| + \frac{||y_k||||F(x_{k+1})||||s_k||}{y_k^T s_k} \quad (47)$$

$$\le ||F(x_{k+1}) + \frac{||y_k||||F(x_{k+1})||||s_k||}{m||s_k||^2} \quad (48)$$

$$\le \left||F(x_{k+1})|\right| + \frac{M||s_k||^2||F(x_{k+1})||}{m||s_k||^2} \quad (49)$$

Inequality (46) follows from Cauchy-Schwartz inequality. From the level set and equation 31 we have;

$$\left||d_{k+1}|\right| \le \left(c + \frac{M}{m}\right)\left||F(x_0)|\right| + \frac{M||F(x_0)||}{m} \quad (50)$$

$$\le \left||F(x_0)|\right| + \frac{M||F(x_0)||}{m} \quad (51)$$

$$\le \left(c + \frac{2M}{m}\right)\left||F(x_0)|\right| = m_2. \quad (52)$$

Therefore, equation 52 shows that equation 43 holds.

We are now going to establish the global convergence of our method, in order to show that under some suitable conditions, there exists an accumulation point of sequence x_k which is a solution of problem in equation 1.

Theorem: Suppose assumption 1 holds and that the sequence $\{ x_k\}$ is generated by the SHCGA algorithm. Also, assume that for all k>0,

$$\alpha_k \geq c \frac{|F(x_k)^T d_k|}{||d_k||^2}, \quad (53)$$

Where c is some positive constant. Then $\{ x_k\}$ converges globally to a solution of problem (1); i.e.,

$$\lim_{k\to\infty} ||F(x_k)|| = 0. \quad (54)$$

Proof: By the boundedness of d_k, we have;

$$\lim_{k\to\infty} \alpha_k ||d_k||^2 = 0. \quad (55)$$

From equations 53 and 55 we have

$$\lim_{k\to\infty} |F(x_k)^T d_k| = 0. \quad (56)$$

From our proposed direction, we have;

$$d_{k+1} = -F(x_{k+1}) + \beta_k s_k \quad (57)$$

Therefore, by multiplying equation 57 by $F(x_{k+1})^T$, we obtain:

$$F(x_{k+1})^T d_{k+1} = -F(x_{k+1})^T F(x_{k+1}) + \beta_k F(x_{k+1})^T s_k. \quad (58)$$

$$||F(x_{k+1})||^2 = -F(x_{k+1})^T d_{k+1} + \beta_k F(x_{k+1})^T s_k \quad (59)$$

$$||F(x_{k+1})||^2 \leq |-F(x_{k+1})^T d_{k+1}| + |\beta_k F(x_{k+1})^T s_k|. \quad (60)$$

$$||F(x_{k+1})||^2 \leq |-F(x_{k+1})^T d_{k+1}| + |\beta_k| || F(x_{k+1})^T || ||s_k||. \quad (61)$$

But (53) implies that;

$$\alpha_k ||d_k||^2 \geq c|F(x_k)^T d_k|, \quad (62)$$

Since $||d_k||$ is bounded and $\lim_{k\to\infty} |F(x_k)^T d_k| = 0$, we have $\lim_{k\to\infty} \alpha_k ||d_k||^2 = 0$. Thus,

$$0 \leq c|F(x_k)^T d_k| \leq \alpha_k ||d_k||^2 \to 0. \quad (63)$$

Then we have;

$$||F(x_k)||^2 \leq |-F(x_k)^T d_k| + |\beta_k| ||F(x_k)^T|| ||s_k|| \to 0. \quad (64)$$

Therefore,

$$\lim_{k\to\infty} \alpha_k ||d_k||^2 = 0. \quad (65)$$

The proof is completed.

3. Numerical Results

In this section, the performance of our method for solving systems of nonlinear equations compared with NHCG method for symmetric nonlinear equations 1 and NHCGP equation 18 are reported.

SHCGA stands for our method and both cases, we set the following:

$\omega_1 = \omega_2 = 10^{-4}, \qquad r = 0.2.$

The codes were written in MATLAB 8.9.0 (R2014a) and run on a personal computer 2.00GHz CPU processor and 3GB RAM memory. We stopped the iterations if the total number of iterations exceeds 1000 or $||F(x_k)|| \leq 10^{-4}$. We tested the three methods using twenty two (22) test problems with different initial starting points (x_0), and dimensions (n-values). We present here some of the benchmark test problems with dimensions 1,000, 10,0000, 20,000, 50,000 and 100,000 respectively used to test our proposed method in this research (i.e SHCGA).

To illustrate the performance of the three methods, a summary of the results is presented in Table 1. Figures 1 and 2 described the performance profile of SHCGA, NHCG and NHCGP Algorithms with respect to the number of iterations for the problems and CPU time (in seconds) for the problems respectively.

Table 1: The summary of numerical results

Item	Algorithms			
	SHCGA	NHCG	NHCGP	Undecided
Total number of problems	130	130	130	
Total number of problems	120	120	120	
Problems solved with less number of iterations	95	10	12	13
Percentage	77.27%	4.54%	6.36%	11.83%
Problems solved with less CPU time	90	10	8	12
Percentage	72.72%	9.09%	7.27%	10.92%

The summarized data shows the number of problems for which method is a winner, in terms of number of iterations and CPU time respectively. The corresponding percentages of number of problems solved by each method are also reported. The summary reported in Table 1 indicates that the SHCGA scheme is a winner with respect to number of iterations and CPU time. The Table 1 shows that, the SHCGA method solves 77.27% (95 out of 120) of the problems with less number of iterations, compared to the NHCG method, which solves 4.54% (10 out of 120) and NHCGP method which solves 6.36% (12 out of 120). The summarized result also shows that both methods solve 13 problems with the same number of iterations, which translates to 11.83% and is reported as undecided. Also, the summary indicates that the SHCGA scheme outperforms the NHCG and NHCGP methods as it solves 72.72% (90 out of 120) of the problems with less CPU time compared to 9.09% (10 out of 120) solved by the NHCG method and 7.27% (8 out of 120) by the NHCGP method. Therefore, it is evident from Figures 1 and 2 and the summarized result in Table 1 that, our method is more effective than the NHCG and NHCGP methods, and therefore, more suitable for solving large-scale system of nonlinear equations.

Figures 1 and 2 show the performance of our method based on the number of iterations and CPU time respectively, which were evaluated using the profiles of Dolan [11].

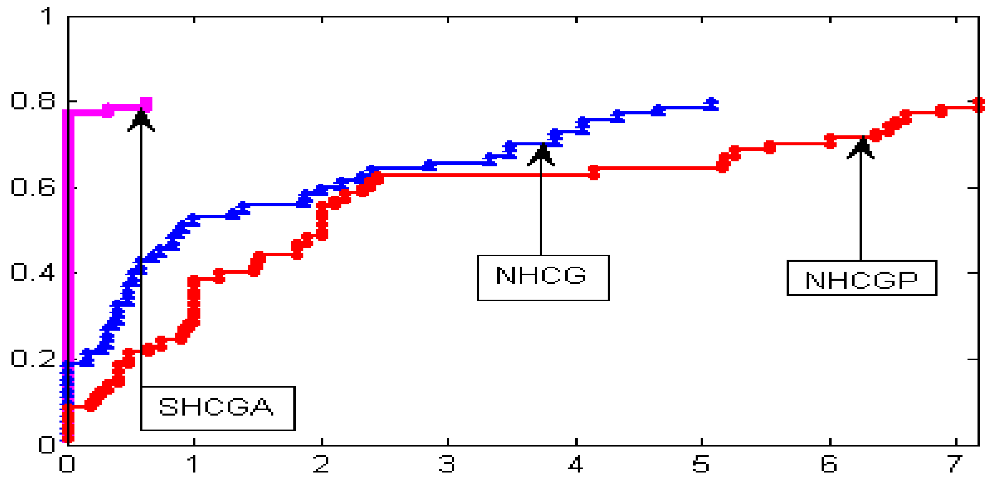

Figure 1: Performance profile of SHCGA, NHCG and NHCGP Algorithms with respect to the number of iterations for the problems

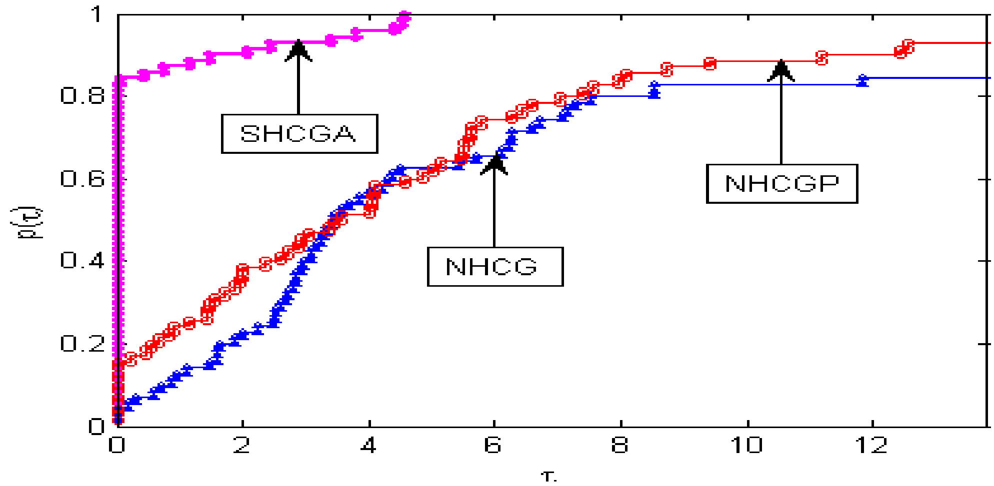

Figure 2: Performance profile of SHCGA, NHCG and NHCGP Algorithms with respect to the CPU time (in seconds) for the problems

For each method, we plot the fraction $p(\tau)$ of the problems for which the method is within a factor τ of the best time. The top curve is the method that solved the most problems in a time that was within a factor τ of the best time. The summary of the numerical results of the three (3) methods are reported in Table 1. The summary of numerical results indicates that the proposed method, i.e SHCGA has minimum number of iterations and CPU time, compared to NHCG and NHCGP respectively. Except for problems 1 and 11 where the number of iterations in SHGCA of large dimension is more than that of NHCG and NHCGP. We can easily see that our claim is fully justified from the table, that is, less CPU time and number of iterations for each test problem with the exception of problems 1 and 11. Furthermore, on the average, $||F(x_k)||$ is too small which signifies that the solution obtained is the true approximation of the exact solution compared to NHCG and NHCGP schemes.

4. Conclusion

In this paper, we presented a new hybrid technics of conjugate gradient algorithm (SHCGA), for solving large-scale system of nonlinear equations and compared its performance with that of (NHCG and NHCGP) methods for symmetric nonlinear equations by performing some numerical experiments. We however proved the global convergence of our proposed method, using a derivative-free line search proposed by Li and Fukushima, and the numerical results show that our method is very effective. This research can be extended to large-scale nonlinear monotone system of equations with applications to signal and image recovery problems.

References

[1]. Abubakar, S. H. and Muhammad, M. W., (2018). An Improved derivative-free method via double direction approach for solving system of Nonlinear equations, *Journal of the Ramanujan Mathematical Society,* 33(1), Pp. 75-89.

[2]. Yu-Hong, D. and Ya-Xiang, Y., (1999). A nonlinear conjugate gradient method with Strong Global Convergence Property, *SIAM Journal of Optimization*, 10, 177-182.

[3]. Zhan-Jun, S. and Jinhua, G., (2008). A new algorithm of nonlinear conjugate gradient method with strong convergence, *Computational and Applied Mathematics*, *27*(1), 9317106

[4]. Li, C., Muhammed, A. H. and Ibrahim, H. Y., (2013).A Modified Conjugate Gradient Method for Unconstrained Optimization, *TELKOMINIKA*, 11(11), 6373-6380.

[5]. Mohammad, A. H., Mustafa, M. and Ibrahim, A. Y., (2014). The Hybrid BFGS-CG Method in Solving Uncontrained optimization Problems*, Hindawi publishing corporation, vol. 2014, pp.6.*

[6]. Mustafa, M., Ibrahim, A. Y. and Mohammed, A. H., (2014). BFGS-Method: A new search direction, sains malaysiana, 43, pp.1591-1597.

[7]. Mohammad, A. H., Mustafa, M. and Ibrahim, A. Y., (2014). The Hybrid BFGS-CG Method in Solving Uncontrained optimization Problems, *Hindawi publishing corporation, vol. 2014, pp.6.*

[8]. Hamed, T. E., Mustafa, M., Ibrahim, A. Y. and Mamat, M., (2019). A new Hybrid Algorithm for Convex Nonlinear Unconstrained Optimization, *Hindawi Journal of applied Mathematics*, 2019, 6-7.

[9]. Waziri, M. Y. and Sabiu. J., (2015). A derivative-free conjugate gradient method and its global convergence for symmetric nonlinear equations, *Journal of mathematics and mathematical sciences*, *2*, 8-10.

[10]. Li, D. and Fukushima, M., (1999). A global and superlinear convergent Gauss-Newton based BFGS Method for symmetric nonlinear equations, *SIAM Journal of Numerical Analysis*, 37, 152-172.

[11]. Sun, W. and Yuan, Y. X., (2006). *An optimization theorem and Methods; nonlinear programming*. Springer, New York, NY, USA.

[12]. Dolan, E, and More J, (2002). A Benchmarking optimization software with performance profiles, *Journal of Mathematical Programming*. *91*(2), 201-213

[13]. Waziri M.Y and Kabiru A. (2019). A Dai-Liao conjugate gradient method via modified secant equation for solving non-linear equations, *Arabian journal of mathematics*, 219, 15-16.

[14]. Yabe, H, and Takano M, (2004). Global convergence properties of new nonlinear conjugate gradient method for unconstrained optimization, *Computational optimization and Application*, *28*, 203-225.

[15]. Jamilu S, Muhammad Y. W, (2017). A New hybrid Dai-Yuan and Hestenes-Stiefel conjugate gradient parameter for solving of non-linear equations, *MAYFEB journal of Mathematics*, 1, 441-755.

[16]. Dauda M.K., Mustafa, M., Muhammad, Y.W., Fadhila, A. and Fatima, S.M.(2016). In-exact Conjugate Gradient Method via SR1 Update for Solving Systems of Non-linear equations, *Far-East Journal of mathematical science (FJMS)*, 100(11), 1787-1804.